理论证明

定理 2.1 证明：用 Z_i 表示第 i 个单元的被抽中的次数，则有

$$\mathrm{E}(Z_i)=np_i,\quad \mathrm{V}(Z_i)=np_i(1-p_i),\quad i=1,2,\cdots,N$$

和

$$\mathrm{Cov}(Z_i,Z_j)=-np_ip_j,\quad i,j=1,2,\cdots,N,\quad i\neq j$$

计算 IHH 估计的偏差为

$$\mathrm{B}\left(\hat{t}_y^*\right)=\mathrm{E}\left(\sum\nolimits_U\frac{Z_iy_i}{np_i^*}-\sum\nolimits_U y_i\right)=\sum\nolimits_U\left(\frac{\mathrm{E}(Z_i)}{np_i^*}-1\right)y_i=\sum\nolimits_{U_2}\left(\frac{p_i}{p_i^*}-1\right)y_i$$

方差为

$$\begin{aligned}\mathrm{V}\left(\hat{t}_y^*\right)&=\sum\nolimits_U\frac{y_i^2\mathrm{V}(Z_i)}{n^2p_i^{*2}}+\sum_{i\neq j}\sum\nolimits_U\frac{y_iy_j\mathrm{Cov}(Z_i,Z_j)}{n^2p_i^*p_j^*}\\&=\sum\nolimits_U\frac{p_i(1-p_i)y_i^2}{np_i^{*2}}-\sum_{i\neq j}\sum\nolimits_U\frac{p_ip_jy_iy_j}{np_i^*p_j^*}\\&=\frac{1}{n}\left[\sum\nolimits_U\frac{p_iy_i^2}{p_i^{*2}}-\left(\sum\nolimits_U\frac{p_iy_i}{p_i^*}\right)^2\right]\end{aligned}$$

因此，IHH 估计的均方误差为

$$\begin{aligned}\mathrm{MSE}\left(\hat{t}_y^*\right)&=\mathrm{B}^2\left(\hat{t}_y^*\right)+\mathrm{V}\left(\hat{t}_y^*\right)\\&=\left[\sum\nolimits_{U_2}\left(\frac{p_i}{p_i^*}-1\right)y_i\right]^2+\frac{1}{n}\sum\nolimits_U\frac{p_iy_i^2}{p_i^{*2}}-\frac{1}{n}\left(\sum\nolimits_U\frac{p_iy_i}{p_i^*}\right)^2\\&=\sum\nolimits_U\frac{n(p_i-p_i^*)^2+p_i(1-p_i)}{np_i^{*2}}y_i^2+\sum_{i\neq j}\sum\nolimits_U\frac{n(p_i-p_i^*)(p_j-p_j^*)-p_ip_j}{np_i^*p_j^*}y_iy_j\end{aligned}$$

又因为 $\mathrm{E}(Z_iZ_j)=\mathrm{Cov}(Z_i,Z_j)+\mathrm{E}(Z_i)\mathrm{E}(Z_j)=n(n-1)p_ip_j$，所以 IHH 估计量均方误差的一个无偏估计为

$$\widehat{\mathrm{MSE}}\left(\hat{t}_y^*\right)=\sum\nolimits_s\frac{n(p_i-p_i^*)^2+p_i(1-p_i)}{n^2p_ip_i^{*2}}y_i^2+\sum_{i\neq j}\sum\nolimits_s\frac{n(p_i-p_i^*)(p_j-p_j^*)-p_ip_j}{n^2(n-1)p_ip_jp_i^*p_j^*}y_iy_j$$

证毕。

定理 2.2 证明：根据条件 $\max_{i\in U}|y_i|<\lambda$ 和 $\lambda_1\leq\min_{i\in U}\{np_i\}<\max_{i\in U}\{np_i\}\leq\lambda_2$，有

$$\left|\mathrm{B}\left(\hat{\bar{t}}_y^*\right)\right|=\left|\frac{1}{N}\sum\nolimits_{U_2}\left(\frac{p_i}{p_i^*}-1\right)y_i\right|\leq\frac{1}{N}\sum\nolimits_{U_2}\left|\frac{np_i-np_i^*}{np_i^*}y_i\right|=O\left(\frac{K}{N}\right)$$

又由于 $p_{(K)}\leq(nK+1)^{-1}$，所以 K 有界，因此 $B^2\left(N^{-1}\hat{t}_y^*\right)=O(n^{-2})$。类似地，对于 IHH 的

方差有

$$\begin{aligned}
\left|\mathrm{V}\left(\hat{t}_y^*\right)\right| &= \left|\frac{1}{N^2 n}\left[\sum_U \frac{p_i y_i^2}{p_i^{*2}} - \left(\sum_U \frac{p_i y_i}{p_i^*}\right)^2\right]\right| \\
&\le \left|\frac{1}{N^2}\sum_U \frac{n p_i y_i^2}{n^2 p_i^{*2}}\right| + \left|\frac{1}{N^2 n}\left(\sum_U \frac{n p_i y_i}{n p_i^*}\right)^2\right| \\
&= O(n^{-1}) + O(n^{-2}) \\
&= O(n^{-1})
\end{aligned}$$

证毕。

定理 2.3 证明：注意到，HH 估计的均方误差（方差）为

$$\mathrm{MSE}\left(\hat{t}_y\right) = \frac{1}{n}\sum_U p_i\left(\frac{y_i}{p_i} - t\right)^2 = \frac{1}{n}\sum_U \frac{y_i^2}{p_i} - \frac{1}{n}\left(\sum_U y_i\right)^2$$

IHH 估计的均方误差为

$$\mathrm{MSE}\left(\hat{t}_y^*\right) = \left[\sum_{U_2}\left(\frac{p_i}{p_i^*} - 1\right) y_i\right]^2 + \frac{1}{n}\sum_U \frac{p_i y_i^2}{p_i^{*2}} - \frac{1}{n}\left(\sum_U \frac{p_i y_i}{p_i^*}\right)^2$$

两个均方误差作差得到

$$\begin{aligned}
\mathrm{MSE}\left(\hat{t}_y\right) - \mathrm{MSE}\left(\hat{t}_y^*\right) &= \frac{1}{n}\sum_U \frac{y_i^2}{p_i} - \frac{1}{n}\sum_U \frac{p_i y_i^2}{p_i^{*2}} - \left[\sum_{U_2}\left(\frac{p_i}{p_i^*} - 1\right) y_i\right]^2 \\
&\quad + \frac{1}{n}\left(\sum_U \frac{p_i y_i}{p_i^*}\right)^2 - \frac{1}{n}\left(\sum_U y_i\right)^2 \\
&= \sum_U \frac{y_i^2}{n p_i} - \sum_U \frac{p_i y_i^2}{n p_i^{*2}} - \left[\sum_{U_2}\left(\frac{p_i}{p_i^*} - 1\right) y_i\right]^2 \\
&\quad + \sum_U \frac{p_i^2 y_i^2}{n p_i^{*2}} + \sum_{i\neq j}\sum_U \frac{p_i y_i p_j y_j}{n p_i^* p_j^*} - \sum_U \frac{y_i^2}{n} - \sum_{i\neq j}\sum_U \frac{y_i y_j}{n} \\
&= \sum_{U_2} \frac{(p_i^{*2} - p_i^2)(1 - p_i)}{n p_i p_i^{*2}} y_i^2 - \left[\sum_{U_2}\left(\frac{p_i}{p_i^*} - 1\right) y_i\right]^2 \\
&\quad + \sum_{i\neq j}\sum_U \left(\frac{p_i y_i p_j y_j}{n p_i^* p_j^*} - \frac{y_i y_j}{n}\right)
\end{aligned}$$

根据柯西不等式，有

$$
\begin{aligned}
\mathrm{MSE}\left(\hat{t}_y\right)-\mathrm{MSE}\left(\hat{t}_y^*\right) \geq & \sum\nolimits_{U_2} \frac{(p_i^{*2}-p_i^2)(1-p_i)}{np_ip_i^{*2}}y_i^2-K\cdot\sum\nolimits_{U_2}\frac{(p_i-p_i^*)^2}{p_i^{*2}}y_i^2 \\
&+\sum_{i\neq j}\sum\nolimits_U\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right) \\
&=\sum\nolimits_{U_2}\frac{(p_i^*-p_i)\left[(1-p_i-nKp_i)p_i^*+(p_i-p_i^2+nKp_i^2)\right]}{np_ip_i^{*2}}y_i^2 \\
&+\sum_{i\neq j}\sum\nolimits_U\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right) \\
&\triangleq A_1+A_2
\end{aligned}
$$

其中，K 为U_2 中的单元个数。显然，当 $p_{(K)}\leq(nK+1)^{-1}$ 时，$1-p_i-nKp_i\geq 0,\ i\in U_2$ 。此时，上式中 $A_1\geq 0$ 恒成立。特别地，当U_2 中单元的入样概率均相等时，等号成立。

为完成整个定理的证明，只需要证明 $A_2=o(N^2n^{-1})$ 即可。注意到，

$$
\begin{aligned}
A_2 &=\sum_{i\neq j}\sum\nolimits_U\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right) \\
&=\sum_{i\neq j}\sum\nolimits_{U_2}\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right)+\sum_{i\in U_1}\sum_{j\in U_2}\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right) \\
&\quad+\sum_{i\in U_2}\sum_{j\in U_1}\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right) \\
&=B_1+B_2+B_3
\end{aligned}
$$

根据条件 $\max\limits_{i\in U}|y_i|<\lambda$ 和 $\lambda_1\leq\min\limits_{i\in U}\{np_i\}<\max\limits_{i\in U}\{np_i\}\leq\lambda_2$，有

$$
\begin{aligned}
|B_1| &=\left|\sum_{i\neq j}\sum\nolimits_{U_2}\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right)\right| \\
&=\left|\sum_{i\neq j}\sum\nolimits_{U_2}\left(\frac{np_inp_j-np_i^*np_j^*}{np_i^*np_j^*}y_iy_j\right)\right| \\
&=O(K^2n^{-1})
\end{aligned}
$$

和

$$
\begin{aligned}
|B_2| &=\left|\sum_{i\in U_1}\sum_{j\in U_2}\left(\frac{p_iy_ip_jy_j}{np_i^*p_j^*}-\frac{y_iy_j}{n}\right)\right| \\
&=\left|\frac{1}{n}\sum_{i\in U_1}\sum_{j\in U_2}\left(\frac{np_inp_j-np_i^*np_j^*}{np_i^*np_j^*}y_iy_j\right)\right| \\
&=O(NKn^{-1})
\end{aligned}
$$

类似于$|B_2|$，可以直接得到$|B_3|=O(NKn^{-1})$。因为 $p_{(K)}\leq(nK+1)^{-1}$，所以$K=O(1)$。因此，$|B_1|$、$|B_2|$和$|B_3|$的阶均为$o(N^2n^{-1})$，故 $A_2=o(N^2n^{-1})$。证毕。

定理 2.4 的证明：注意到,

$$\hat{\bar{t}}_y^* = \frac{1}{n}\sum\nolimits_s \frac{y_i}{Np_i^*} \triangleq \frac{1}{n}\sum\nolimits_s \eta_i$$

* MERGEFORMAT (1)

由于每次抽样都是独立进行的，因此$\{\eta_i\}_{i\in s}$是相互独立的。根据定理 2.1 和定理 2.2 知

$$\sum\nolimits_s \mathrm{E}\left(n^{-1/2}\eta_i\right) = \sqrt{n}\cdot\mathrm{E}\left(\hat{\bar{t}}_y^*\right) = \bar{t}_y + O(n^{-1})$$ * MERGEFORMAT (2)

且

$$\Sigma \triangleq \sum\nolimits_s \mathrm{V}\left(n^{-1/2}\eta_i\right) = n\cdot\mathrm{V}\left(N^{-1}\hat{t}_y^*\right) = \frac{1}{N^2}\left[\sum\nolimits_U \frac{p_i y_i^2}{p_i^{*2}} - \left(\sum\nolimits_U \frac{p_i y_i}{p_i^*}\right)^2\right] = O(1)$$ * MERGEFORMAT (3)

此外，对任意的$\varepsilon>0$，都存在$\delta>0$，使得

$$\begin{aligned}
\sum\nolimits_s \mathrm{E}\left[\left|n^{-1/2}\eta_i\right|^2 \mathrm{I}\left(|\eta_i| > n^{1/2}\varepsilon\right)\right] &\le \sum\nolimits_s \mathrm{E}\left[\frac{\left|n^{-1/2}\eta_i\right|^{2+\delta}}{\varepsilon^\delta}\mathrm{I}\left(|\eta_i| > n^{1/2}\varepsilon\right)\right] \\
&\le \sum\nolimits_s \mathrm{E}\left(\frac{\left|n^{-1/2}\eta_i\right|^{2+\delta}}{\varepsilon^\delta}\right) \\
&= \frac{1}{n^{1+\delta/2}\varepsilon^\delta}\mathrm{E}\left(\sum\nolimits_s |\eta_i|^{2+\delta}\right) \\
&= \frac{1}{n^{1+\delta/2}\varepsilon^\delta}\sum\nolimits_U \left(np_i|\eta_i|^{2+\delta}\right) \\
&= \frac{n^{1+\delta}}{N^{1+\delta}n^{\delta/2}\varepsilon^\delta}\cdot\frac{1}{N}\sum\nolimits_U\left(np_i\left|\frac{y_i}{np_i^*}\right|^{2+\delta}\right) \\
&= o(1)
\end{aligned}$$

最后一个等号是根据条件$\max_{i\in U}|y_i| < \lambda$和$\lambda_1 \le \min_{i\in U}\{np_i\} < \max_{i\in U}\{np_i\} \le \lambda_2$获得的。因此，根据 Lindeberg-Feller 中心极限定理（见文献 Vaart（1998）中命题 2.27）有

$$\sum\nolimits_s\left[n^{-1/2}\eta_i - E\left(n^{-1/2}\eta_i\right)\right] \xrightarrow{d} N(0,\Sigma)$$

进一步，结合式（1）～（3）整理以上分布得到$\sqrt{n}\cdot\left(\hat{\bar{t}}_y^* - \bar{t}_y\right)\xrightarrow{d} N(0,\Sigma)$。证毕。

定理 3.1 证明：注意到，

$$\begin{aligned}
\mathrm{MSE}\left(\hat{t}_{sy}^*\right) &= \mathrm{E}\left(\hat{t}_{sy}^* - t\right)^2 \\
&= \mathrm{E}\left[\sum_{h=1}^{L}\left(\hat{t}_h^* - t_h\right)\right]^2 \\
&= \sum_{h=1}^{L}\mathrm{MSE}\left(\hat{t}_h^*\right) + \sum_{h\ne l}^{L}\sum \mathrm{E}\left[\left(\hat{t}_h^* - t_h\right)\left(\hat{t}_l^* - t_l\right)\right]
\end{aligned}$$

由于层之间的抽样是独立进行的，因而根据定理 2.2，有

$$\frac{1}{N_h N_l}\mathrm{E}\left[\left(\hat{t}_h^* - t_h\right)\left(\hat{t}_l^* - t_l\right)\right] = \frac{1}{N_h N_l}\mathrm{E}\left(\hat{t}_h^* - t_h\right)\mathrm{E}\left(\hat{t}_l^* - t_l\right) = O\left(\frac{1}{n_h n_l}\right)$$

因此，根据条件 $n_h/n \to \lambda_h$ 可知

$$\mathrm{MSE}\left(\hat{t}_{sy}^{*}\right)=\sum_{h=1}^{L}\mathrm{MSE}\left(\hat{t}_{h}^{*}\right)+o(N^2n^{-1})$$

进一步，可推出均方误差的渐近无偏估计为

$$\widehat{\mathrm{MSE}}\left(\hat{t}_{sy}^{*}\right)=\sum_{h=1}^{L}\widehat{\mathrm{MSE}}\left(\hat{t}_{h}^{*}\right)+o(N^2n^{-1})$$

其中，$\widehat{\mathrm{MSE}}\left(\hat{t}_{h}^{*}\right)$表示第 h 层 IHH 估计的无偏均方误差估计。证毕。

定理 3.2 证明：根据定理 2.3，有

$$\sum_{h=1}^{L}\mathrm{MSE}\left(\hat{\hat{t}}_{h}^{*}\right)„ \sum_{h=1}^{L}\mathrm{MSE}\left(\hat{\hat{t}}_{h}\right)+o(n^{-1})$$

再结合式（4）～（5），得到

$$\mathrm{MSE}\left(\hat{\hat{t}}_{sy}^{*}\right)„ \mathrm{MSE}\left(\hat{\hat{t}}_{sy}\right)+o(n^{-1})$$

证毕。

定理 4.1 证明：注意到，

$$\begin{aligned}\mathrm{B}\left(\hat{t}_{cy}^{*}\right)&=\mathrm{E}_1\mathrm{E}_2\left(\frac{N}{n}\sum\nolimits_s\hat{t}_i^{*}-t\right)\\&=\mathrm{E}_1\left[\frac{N}{n}\sum\nolimits_s\mathrm{E}_2(\hat{t}_i^{*})-t\right]\\&=\sum_{i=1}^{N}\left[\mathrm{E}_2(\hat{t}_i^{*})-t_i\right]\\&=\sum_{i=1}^{N}\sum_{j=1}^{N_i}\left(\frac{p_{ij}}{p_{ij}^{*}}-1\right)y_{ij}\end{aligned}$$

且

$$\begin{aligned}\mathrm{V}\left(\hat{t}_{cy}^{*}\right)&=\mathrm{V}_1\mathrm{E}_2\left(\frac{N}{n}\sum\nolimits_s\hat{t}_i^{*}\right)+\mathrm{E}_1\mathrm{V}_2\left(\frac{N}{n}\sum\nolimits_s\hat{t}_i^{*}\right)\\&=\mathrm{V}_1\left[\frac{N}{n}\sum\nolimits_s\mathrm{E}_2(\hat{t}_i^{*})\right]+\frac{N^2}{n^2}\mathrm{E}_1\left[\sum\nolimits_s\mathrm{V}_2(\hat{t}_i^{*})\right]\\&=\frac{N(N-n)}{n(N-1)}\sum_{i=1}^{N}\left(t_i^{*}-\overline{\overline{t}}^{*}\right)^2+\frac{N}{n}\sum_{i=1}^{N}\mathrm{V}_2(\hat{t}_i^{*})\end{aligned}$$

这里 $t_i^{*}=\sum_{j=1}^{N_i}\frac{p_{ij}}{p_{ij}^{*}}y_{ij}$，$\overline{\overline{t}}^{*}=\frac{1}{N}\sum_{i=1}^{N}t_i^{*}$ 且 $\mathrm{V}_2(\hat{t}_i^{*})=\frac{1}{n_i}\sum_{j=1}^{N_i}\frac{p_{ij}y_{ij}^2}{p_{ij}^{*2}}-\frac{1}{n_i}\left(\sum_{j=1}^{N_i}\frac{p_{ij}y_{ij}}{p_{ij}^{*}}\right)^2$。因此，有

$$\mathrm{MSE}\left(\hat{t}_{cy}^{*}\right)=\left[\sum_{i=1}^{N}\sum_{j=1}^{N_i}\left(\frac{p_{ij}}{p_{ij}^{*}}-1\right)y_{ij}\right]^2+\frac{N(N-n)}{n(N-1)}\sum_{i=1}^{N}\left(t_i^{*}-\overline{\overline{t}}^{*}\right)^2+\frac{N}{n}\sum_{i=1}^{N}\mathrm{V}_2(\hat{t}_i^{*}) \backslash * \text{ MERGEFORMAT } (7)$$

证毕。

定理 4.2 证明：记 K_i 为第 i 个群中的修正点，则根据定义 2.1 可知 K_i 有界。进一步，可推

出

$$\left[\sum_{i=1}^{N}\sum_{j=1}^{N_i}\left(\frac{p_{ij}}{p_{ij}^{*}}-1\right)y_{ij}\right]^2 „ \left[\sum_{i=1}^{N}\sum_{j=1}^{N_i}\left|\left(\frac{n_i p_{ij}-n_i p_{ij}^{*}}{n_i p_{ij}^{*}}\right)y_{ij}\right|\right]^2 „ C\left[\sum_{i=1}^{N}K_i\right]^2=O(N^2) \text{ * MERGEFORMAT (8)}$$

根据定理 2.2 和定理 2.3，有

$$\frac{N}{n}\sum_{i=1}^{N}\mathrm{V}_2(\hat{t}_i)-\frac{N}{n}\sum_{i=1}^{N}\mathrm{V}_2(\hat{t}_i^{*})=\frac{N}{n}\sum_{i=1}^{N}\left[\mathrm{V}_2(\hat{t}_i)-\mathrm{V}_2(\hat{t}_i^{*})\right]=o\left(\frac{N}{n}\sum_{i=1}^{N}\frac{N_i^2}{n_i}\right) \text{ * MERGEFORMAT (9)}$$

注意到

$$\begin{aligned}
N\left|\bar{\bar{t}}^{*2}-\bar{\bar{t}}^{2}\right| &„ N\left|\bar{\bar{t}}^{*}-\bar{\bar{t}}\right|\left|\bar{\bar{t}}^{*}+\bar{\bar{t}}\right| \\
&=\left|\sum_{i=1}^{N}\left(t_i^{*}-t_i\right)\right|\left|\frac{1}{N}\sum_{i=1}^{N}\left(t_i^{*}+t_i\right)\right| \\
&=\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_i}\frac{n_i p_{ij}-n_i p_{ij}^{*}}{n_i p_{ij}^{*}}y_{ij}\right|\left|\sum_{i=1}^{N}\sum_{j=1}^{N_i}\frac{p_{ij}+p_{ij}^{*}}{p_{ij}^{*}}y_{ij}\right| \\
&„ C\left(\frac{1}{N}\sum_{i=1}^{N}K_i\right)\left(\sum_{i=1}^{N}N_i\right)=O\left(\sum_{i=1}^{N}N_i\right)
\end{aligned}$$

因此

$$\begin{aligned}
\sum_{i=1}^{N}\left(t_i-\bar{\bar{t}}\right)^2-\sum_{i=1}^{N}\left(t_i^{*}-\bar{\bar{t}}^{*}\right)^2 &=\sum_{i=1}^{N}\left(t_i^2-t_i^{*2}\right)+N\left(\bar{\bar{t}}^{*2}-\bar{\bar{t}}^{2}\right) \\
&=\sum_{i=1}^{N}\left(1-\frac{p_{ij}^2}{p_{ij}^{*2}}\right)y_{ij}^2+O\left(\sum_{i=1}^{N}N_i\right)
\end{aligned} \text{ * MERGEFORMAT (10)}$$

结合式（6）～（10），比较 THH 和 TIHH 的均方误差得到

$$\begin{aligned}
\mathrm{MSE}\left(\hat{t}_{cy}\right)-\mathrm{MSE}\left(\hat{t}_{cy}^{*}\right) &=\frac{N(N-n)}{n(N-1)}\left[\sum_{i=1}^{N}\left(t_i-\bar{\bar{t}}\right)^2-\sum_{i=1}^{N}\left(t_i^{*}-\bar{\bar{t}}^{*}\right)^2\right] \\
&\quad+\frac{N}{n}\sum_{i=1}^{N}\left[\mathrm{V}_2(\hat{t}_i)-\mathrm{V}_2(\hat{t}_i^{*})\right]-\left[\sum_{i=1}^{N}\sum_{j=1}^{N_i}\left(\frac{p_{ij}}{p_{ij}^{*}}-1\right)y_{ij}\right]^2 \\
&=\frac{N(N-n)}{n(N-1)}\sum_{i=1}^{N}\left(1-\frac{p_{ij}^2}{p_{ij}^{*2}}\right)y_{ij}^2+O\left(\sum_{i=1}^{N}N_i\right) \\
&\quad+o\left(\frac{N}{n}\sum_{i=1}^{N}\frac{N_i^2}{n_i}\right)+O(N^2) \\
&=\frac{N(N-n)}{n(N-1)}\sum_{i=1}^{N}\left(1-\frac{p_{ij}^2}{p_{ij}^{*2}}\right)y_{ij}^2+O\left(\sum_{i=1}^{N}N_i\right)
\end{aligned}$$

这里最后一个等式成立是由于 $n/N\to\lambda_0$ 且 $n_i/N_i\to\lambda_i$。再根据 $p_{ij}^{*}-p_{ij}\ldots 0$，有

$$\mathrm{MSE}\left(\hat{t}_{cy}^{*}\right)„ \mathrm{MSE}\left(\hat{t}_{cy}\right)+O\left(\sum_{i=1}^{N}N_i\right)$$

证毕。